TRINITY COLLEGE

IN NOMINE DOMINI

Semester Two Examination, 2023

Question/Answer booklet

MATHEMATICS METHODS UNITS 3&4

Section One: Calculator-free

SOLUTIONS

WA student number: In figures

In words ______________________

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Your name ______________________

Number of additional answer booklets used (if applicable):

Time allowed for this section

Reading time before commencing work: five minutes
Working time: fifty minutes

Materials required/recommended for this section

To be provided by the supervisor

This Question/Answer booklet
Formula sheet

To be provided by the candidate

Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: nil

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of examination
Section One: Calculator-free	7	7	50	52	35
Section Two: Calculator-assumed	12	12	100	98	65
				Total	100

Instructions to candidates

1. The rules for the conduct of examinations are detailed in the school handbook. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer booklet preferably using a blue/black pen. Do not use erasable or gel pens.

3. You must be careful to confine your answers to the specific question asked and to follow any instructions that are specific to a particular question.

4. Show all your working clearly. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

5. It is recommended that you do not use pencil, except in diagrams.

6. Supplementary pages for planning/continuing your answers to questions are provided at the end of this Question/Answer booklet. If you use these pages to continue an answer, indicate at the original answer where the answer is continued, i.e. give the page number.

7. The Formula sheet is not to be handed in with your Question/Answer booklet.

Section One: Calculator-free **35% (52 Marks)**

This section has **seven** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time: 50 minutes.

Question 1 **(8 marks)**

(a) Solve the following equations for x.

(i) $e^x = 7$. (1 mark)

Solution
$x = \log_e 7 = \ln 7 = \dfrac{\log_a 7}{\log_a e}$
Specific behaviours
✓ any correct form

(ii) $\log_2(x-7) + \log_2(x+7) = 5$. (3 marks)

Solution
$\log_2\big((x-7)(x+7)\big) = 5$ $(x-7)(x+7) = 2^5$ $x^2 - 49 = 32$ $x^2 = 81$ $x = 9$ *NB positive root only as $x = -9$ does not satisfy original equation.*
Specific behaviours
✓ simplifies LHS using log laws ✓ eliminates logs ✓ correct solution

(b) Function f is defined by $f(x) = \log_e(x+5) - 2$. Determine

(i) the equation of the asymptote of the graph of $y = f(x)$. (1 mark)

Solution
$x = -5$
Specific behaviours
✓ correct equation

(ii) the coordinates of the point on the graph of $y = f(x)$ that has a slope of $\frac{1}{2}$. (3 marks)

Solution
$f'(x) = \dfrac{1}{x+5}$ $\dfrac{1}{x+5} = \dfrac{1}{2} \Rightarrow x = -3$ $f(-3) = \ln(2) - 2$ Point is at $(-3, \ln(2) - 2)$.
Specific behaviours
✓ correct $f'(x)$ ✓ correct x-coordinate ✓ correct coordinates

$\left(-3, \ln\left(\frac{2}{e^2}\right)\right)$

Question 2 **(7 marks)**

(a) Determine $\frac{dy}{dx}$ when

(i) $y = e^{\sin(x+4)}$. (2 marks)

Solution
$\frac{dy}{dx} = \cos(x+4)\, e^{\sin(x+4)}$
Specific behaviours
✓ differentiates trig term correctly ✓ correct derivative

(ii) $y = \int_2^x \ln(t^2 - 3t)\, dt$. (1 mark)

Solution
$\frac{dy}{dx} = \ln(x^2 - 3x)$
Specific behaviours
✓ correct derivative

(b) Determine $\frac{d}{dx}(x\ln(3x))$. (2 marks)

Solution
$\frac{dy}{dx} = \ln(3x) + x\left(\frac{3}{3x}\right) = \ln(3x) + 1$
Specific behaviours
✓ indicates use of product rule ✓ correct derivative

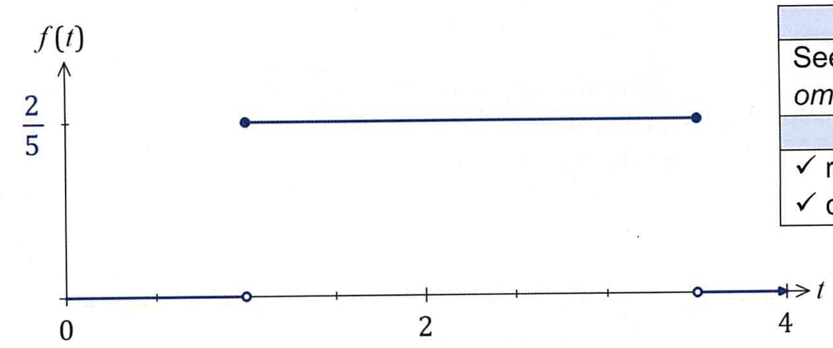
(c) Hence, or otherwise, determine $\int (\ln(3x) + 5)\, dx$. (2 marks)

Solution
$\ln(3x) + 5 = \ln(3x) + 1 + 4$ $\int (\ln(3x) + 5)\, dx = \int (\ln(3x) + 1)\, dx + \int 4\, dx$ $= x\ln(3x) + 4x + c$
Specific behaviours
✓ indicates appropriate use of previous result ✓ correct antiderivative

Question 3 **(7 marks)**

The random variable T is the time in hours that a plumber takes to replace a hand basin in a bathroom and is uniformly distributed between 1 and 3.5 hours. The mean and standard deviation of T are 2.25 and 0.7 hours respectively.

(a) Sketch the probability density function of T on the axes below. (2 marks)



Solution
See graph. (*No penalty for omitting $y = 0$ rays*)
Specific behaviours
✓ reasonable line segment
✓ correct scales

(b) Determine

(i) $P(T \geq 3)$. (1 mark)

Solution
$P(T \geq 3) = \dfrac{3.5-3}{3.5-1} = \dfrac{0.5}{2.5} = \dfrac{1}{5}$
Specific behaviours
✓ correct probability

(ii) the value of the constant k when $P(T > 2.5 \mid T < k) = 0.2$. (2 marks)

Solution
$P(T > 2.5 \mid T < k) = \dfrac{k-2.5}{k-1} = 0.2$ $k - 2.5 = 0.2k - 0.2$ $0.8k = 2.3$ $k = \dfrac{23}{8}$
Specific behaviours
✓ formulates correct equation for k
✓ correct value of k

The random variable C is the amount in dollars that the plumber charges for labour to replace a hand basin in a bathroom. The plumber charges \$60 per hour plus a fixed call-out fee of \$55.

(c) Determine the mean and standard deviation of C. (2 marks)

Solution
$C = 60T + 55$ $\bar{C} = 60 \times 2.25 + 55 = 120 + 15 + 55 = \$190$ $\text{sd}_C = 60 \times 0.7 = \$42$
Specific behaviours
✓ correct mean
✓ correct standard deviation

Question 4 **(8 marks)**

A tank initially contains 23 L of water. Let $V(t)$ be the volume, in litres, of water in the tank t seconds after it is ruptured, so that

$$V'(t) = -\frac{8t}{t^2+3}, \qquad 0 \le t \le 30.$$

Determine

(a) $V'(2)$. (1 mark)

Solution

$$V'(2) = -\frac{8(2)}{2^2+3} = -\frac{16}{7}\ \text{L/s}$$

Specific behaviours

✓ correct value

(b) $V''(2)$. (3 marks)

Solution

$$V''(t) = -\frac{8(t^2+3) - 8t(2t)}{(t^2+3)^2} = -\frac{24-8t^2}{(t^2+3)^2}$$

$$V''(2) = -\frac{24-8(2)^2}{(2^2+3)^2}$$
$$= -\frac{24-32}{49}$$
$$= \frac{8}{49}\ \text{L/s}^2$$

Specific behaviours

✓ indicates correct use of quotient rule
✓ correct derivative
✓ correct value

$\frac{8t^2-24}{(t^2+3)^2}$

(c) $V(2)$. (4 marks)

Solution

$$V(2) = 23 + \int_0^2 -\frac{8t}{t^2+3}\,dt$$
$$= 23 - 4\int_0^2 \frac{2t}{t^2+3}\,dt$$
$$= 23 - 4[\ln(t^2+3)]_0^2$$
$$= 23 - 4(\ln(7) - \ln(3))$$
$$= 23 - 4\ln\left(\frac{7}{3}\right)\ \text{L}$$

Specific behaviours

✓ integrates $V'(t)$ correctly
✓ simplifies definite integral
✓ indicates use of initial volume
✓ correct volume

$4\ln\left(\frac{3}{7}\right) + 23$

Question 5 **(7 marks)**

A hydraulic press exerts a force of F kilonewtons during the 6 seconds it takes to compress a pellet. Initially it exerts no force and t seconds after it is started, the rate of change of force is given by

$$\frac{dF}{dt} = 3\pi \sin\left(\frac{\pi t}{5}\right).$$

(a) Use calculus to show that the force exerted by the press is increasing at the greatest rate 2.5 seconds after it starts. (3 marks)

Solution

Force is increasing when $t = 2.5$ since

$$\frac{dF}{dt} = 3\pi \sin\left(\frac{\pi(2.5)}{5}\right) = 3\pi \text{ kN/s}$$

Rate of increase is greatest when $F''(t) = 0$:

$$\frac{d^2F}{dt^2} = \frac{3\pi^2}{5}\cos\left(\frac{\pi t}{5}\right)$$

$$\frac{d^2F}{dt^2} = 0 \Rightarrow \cos\left(\frac{\pi t}{5}\right) = 0, 0 \le t \le 6, \quad \frac{\pi t}{5} = \frac{\pi}{2}, \qquad t = 2.5 \text{ s}$$

Specific behaviours

✓ shows force is increasing
✓ correct second derivative
✓ clearly derives $t = 2.5$

(b) Determine an expression for the force exerted by the hydraulic press at time t. (2 marks)

Solution

$$F(t) = \int 3\pi \sin\left(\frac{\pi t}{5}\right) dt$$

$$= -15\cos\left(\frac{\pi t}{5}\right) + c$$

$$(0,0) \Rightarrow -15\cos(0) + c = 0, \qquad c = 15$$

$$F(t) = -15\cos\left(\frac{\pi t}{5}\right) + 15$$

Specific behaviours

✓ correct antiderivative
✓ obtains constant of integration and writes function

(c) Determine the maximum force exerted by the hydraulic press during the 6 seconds that it operates. (2 marks)

Solution

$$\frac{dF}{dt} = 0 \Rightarrow \sin\left(\frac{\pi t}{5}\right) = 0, \qquad \frac{\pi t}{5} = \pi, \qquad t = 5 \text{ s}$$

$$F(5) = 15 - 15\cos\left(\frac{\pi(5)}{5}\right)$$

$$= 15 - 15 \times (-1)$$

$$= 30 \text{ kN}$$

Specific behaviours

✓ indicates time when force is maximum
✓ correct maximum force

Question 6 **(8 marks)**

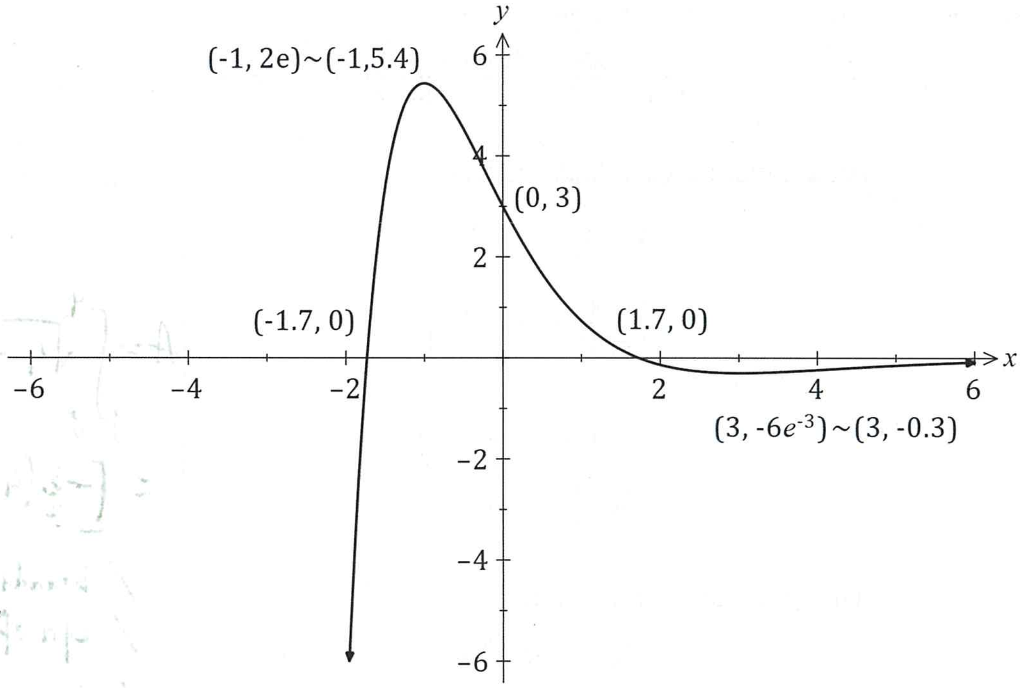
Let $f(x) = \dfrac{3 - x^2}{e^x}$, so that $f'(x) = \dfrac{(x+1)(x-3)}{e^x}$ and $f''(x) = \dfrac{1 + 4x - x^2}{e^x}$.

(a) Determine the nature and location of all stationary points of $f(x)$. (3 marks)

Solution
Stationary points when $f'(x) = 0 \Rightarrow x = -1, x = 3$. $f(-1) = 2e$ and so, at $(-1, 2e)$ there is a maximum since $f''(-1) < 0$. $f(3) = -6 \div e^3$ and so, at $(3, -6e^{-3})$ there is a minimum since $f''(3) > 0$.
Specific behaviours
✓ obtains location of one stationary point ✓ obtains location of second stationary point ✓ correctly uses $f''(x)$ to describe nature of stationary points

(b) Sketch the graph of $y = f(x)$. (5 marks)

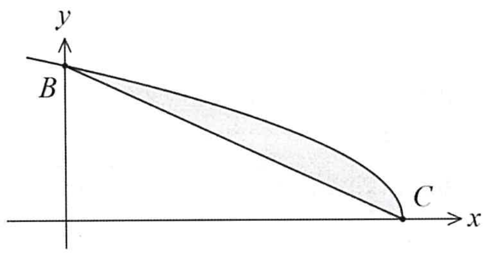
Some approximations that you may assume are $e \approx 2.7, e^2 \approx 7.4, e^3 \approx 20$, $f(x) \geq 0$ only when $-1.7 \leq x \leq 1.7$ and $f''(x) \geq 0$ only when $-0.2 \leq x \leq 4.2$.



Solution
Max: $(-1, 2e) \approx (-1, 5.4)$, and min: $(3, -6e^{-3}) \approx (3, -0.3)$. Roots: $x = -1.7$ and $x = 1.7$. y-intercept: $f(0) = 3$. Points of inflection when $x = -0.2$ and $x = 4.2$. As $x \to \infty, f(x) \to 0$ and as $x \to -\infty, f(x) \to -\infty$.
Specific behaviours
✓ adds appropriate scales to axes ✓ correct axes intercepts ✓ correctly plots minimum and maximum ✓ concave up curve for $-0.2 \leq x \leq 4.2$ ✓ concave down curve elsewhere

Question 7 **(7 marks)**

The graph of the curve $y = \sqrt{4-x}$ is shown to the right together with the chord BC that joins the points of intersection of the curve with the axes.



(a) Determine the slope of the curve at B. (2 marks)

Solution
$$y' = -\frac{1}{2\sqrt{4-x}}$$ $$f'(0) = -\frac{1}{2(2)} = -\frac{1}{4}$$
Specific behaviours
✓ correct y' ✓ correct value of slope

(b) Determine the area of the shaded region. (5 marks)

Solution
$x = 0 \Rightarrow y = 2, \qquad y = 0 \Rightarrow x = 4$ Area under curve in first quadrant: $$A = \int_0^4 (4-x)^{\frac{1}{2}}\,dx = \left[-\frac{2(4-x)^{\frac{3}{2}}}{3}\right]_0^4 = 0 - \left(-\frac{2}{3}(4)^{\frac{3}{2}}\right) = \frac{16}{3} = 5\frac{1}{3}$$ Triangular area under chord: $$A = \frac{1}{2}(4)(2) = 4$$ Area of shaded region: $$A = 5\frac{1}{3} - 4 = 1\frac{1}{3} = \frac{4}{3} \text{ sq units}$$
Specific behaviours
✓ writes correct definite integral ✓ correctly antidifferentiates ✓ correct area under curve ✓ correct area under chord ✓ correct shaded region

$$A = \int_0^4 \sqrt{4-x} - \left(-\tfrac{1}{2}x + 2\right)$$
$$= \left[-\frac{2}{3}(4-x)^{3/2} + \frac{x^2}{4} - 2x\right]_0^4$$

✓ boundaries
✓ eqn of line
✓ integral
✓ correct anti-d
✓ correct area

End of questions